TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 243, September 1978

DEGREES OF IRREDUCIBLE CHARACTERS OF (B, N)-PAIRS OF TYPES E_6 AND E_7 [1]

BY

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ABSTRACT. Let G be a finite (B, N)-pair whose Coxeter system is of type E_6 or E_7. Let 1_B^G be the permutation character of the action of G on the left cosets of the Borel subgroup B in G. In this paper we give the character degrees of the irreducible constituents of 1_B^G.

0. Introduction. Let G be a finite (B, N)-pair of type E_6 or E_7, and let B be a Borel subgroup of G. In this paper we outline techniques that can be used to compute the degrees of the irreducible constituents of the permutation character 1_B^G. We remark that the methods described herein were also used to compute the degrees of some of the irreducible constituents of 1_B^G, where $G = F_4(q)$ or $G = {}^2E_6(q^2)$. These degrees can be found in [3, p. 157]. The degrees of the remaining constituents of 1_B^G, $G = F_4(q)$ or $G = {}^2E_6(q^2)$ were computed in [19] by different techniques. Since the degrees of the irreducible constituents of 1_B^G are known if G is of classical type (see [2], [11]) or if G is rank two (see [13]), only the case $G = E_8(q)$ remains.

Here is a survey of the contents of this paper. In §1 we define a system $\mathcal{S}$ of finite groups with (B, N)-pairs, and we define the corresponding generic algebra A. Following [8] we show how the irreducible characters of A establish a one-to-one correspondence between the irreducible constituents of 1_B^G and the irreducible characters of the Weyl group W.

In §2 we define the generic degree, d_ψ, corresponding to any irreducible character ψ of A. The generic degrees are important in that they specialize to the degrees of the irreducible constituents of 1_B^G. We then describe how the generic degrees d_ψ can be computed by solving an underdetermined system of linear equations in the unknowns d_ψ, subject to various side constraints. The most important side constraint is the result of Benson and Curtis [1] that

Received by the editors April 29, 1977.

AMS (MOS) subject classifications (1970). Primary 20C15; Secondary 16A25.

Key words and phrases. System of (B, N) pairs, irreducible characters, generic algebras, generic degrees, specializations.

[1]This paper constitutes part of a Ph.D. thesis submitted to the Graduate Faculty of the University of Arizona in the fall of 1975. It was written under the supervision of Professor L. C. Grove.

states that the generic degrees satisfy $d_\psi \in \mathbf{Q}[u]$, where u is an indeterminate over the rational numbers $\mathbf{Q}$.

In §3 we return to the cases $G = E_6(q)$ or $G = E_7(q)$ and refine the side constraints discussed in §2. Application of these constraints uniquely determines all generic degrees considered with the exception of those corresponding to the two irreducible characters of degree 512 of the Weyl group of type E_7. We conclude the section with a discussion of how these two exceptional degrees can be computed.

§4 contains the main results of this paper, viz., the tables giving the degrees d_ψ where $G = E_6(q)$ and $G = E_7(q)$.

§5, an appendix, contains tables of decomposition multiplicities necessary for setting up the linear equations mentioned in §2.

1. Irreducible constituents of 1_B^G. The following definition is from [**8**, §5].

Definition 1.1. A system $\mathcal{S}$ of (B, N)-pairs of type (W, R) consists of a finite Coxeter system (W, R), an infinite set $\mathcal{CP}$ of prime powers q called *characteristic powers*, a set $\{c_r, r \in R\}$ of positive integers, and for each $q \in \mathcal{CP}$, a finite group $G = G(q)$ with a (B, N)-pair having (W, R) as its Coxeter system, such that

(i) $c_r = c_s$, $r, s \in R$, if r and s are conjugate in W, and

(ii) for each group $G(q) \in \mathcal{S}$, the index parameters $\operatorname{ind}_B(r)$, are given by $\operatorname{ind}_B(r) = q^{c_r}$, where $\operatorname{ind}_B(r) = [B\colon B \cap rBr]$, $r \in R$.

If G is any (B, N)-pair of type (W, R), $|R| > 2$, then from the classification of J. Tits [**21**], G belongs to some system $\mathcal{S}$ of (B, N)-pairs of type (W, R). Moreover Tits' classification also shows that the index parameters q^{c_r} correspond to those of a Chevalley (or twisted) group of type (W, R). For the possibilities of q^{c_r}, see, e.g., R. Carter's book [**5**].

Now let k be an algebraically closed field of characteristic zero and let $H_k(G, B)$ be the Hecke algebra (or centralizer ring) of G relative to B. Then it is well known [7] that the irreducible k-character constituents of 1_B^G are in a natural one-to-one correspondence with the irreducible characters of $H_k(G, B)$. Moreover, it is shown in [**8**] that $H_k(G, B) \cong kW$, the k-group algebra of W, and so there is a one-to-one correspondence between the irreducible characters of W and the irreducible constituents of 1_B^G. In order to make this correspondence more precise we recall the *generic algebra* $A = A(u)$ for a Coxeter system (W, R), as defined by Tits in [**4**].

Let u be an indeterminate over k and let D be the polynomial ring $D = k[u]$, $K = k(u)$ and let $\bar{K}$ be an algebraic closure of K. Then $A(u)$ is the associative D-algebra with identity having free basis $\{a_w\colon w \in W\}$ over D and multiplication determined by

$$a_r a_w = a_{rw}, \quad \text{if } r \in R, w \in W \text{ and } l(rw) > l(w),$$

$$a_r a_w = u^{c_r} a_{rw} + (u^{c_r} - 1)a_w, \quad \text{if } r \in R, w \in W, \text{ and } l(rw) < l(w), \tag{1.2}$$

where l is the usual length function on W.

Now let f: $D \to k$ be a homomorphism. Then we can define the *specialized algebra* over k,

$$A_f = k \otimes_D A,$$

which has as a k-basis $\{a_{wf}: w \in W\}$, where $a_{wf} = 1 \otimes a_w$, $w \in W$. The map f: $\Sigma u_w a_w \mapsto \Sigma f(u_w) a_{wf}$, $u_w \in D$, can be viewed as a homomorphism of algebras over D, if we view A_f as a D-algebra, with ra defined to be $f(r)a$, $r \in D$, $a \in A_f$.

If f_1: $D \to k$ is defined by setting $f_1(u) = 1$, then from (1.2) we conclude that

$$A(1) = A_{f_1} \cong kW.$$

If f_q: $D \to k$ is defined by setting $f_q(u) = q$, $q \in \mathcal{C}\mathcal{P}$, then from (1.2) and the work of N. Iwahori in **[12]** we conclude that

$$A(q) = A_{f_q} \cong H_k(G, B).$$

Throughout this paper, f_1 and f_q will denote the above specializations.

The following appears in **[8**, Proposition 7.1] and establishes one-to-one correspondences between the irreducible characters of $A^{\bar{K}} \cong \bar{K} \otimes_D A$ with those of $H_k(G, B)$ and with those of kW.

PROPOSITION 1.3. *Let D^* be the integral closure of D in $\bar{K}$. If ψ is an irreducible character of $A^{\bar{K}}$ then $\psi(a_w) \in D^*$ for all $w \in W$. If f_q^* and f_1^* are extensions of f_q and f_1 to D^* then the characters ψ_{f_q} and ψ_{f_1} defined by $\psi_{f_q}(a_{wf_q}) = f_q^* \psi(a_w)$ and $\psi_{f_1}(a_{wf_1}) = f_1^* \psi(a_w)$ are irreducible characters of $H_k(G, B)$ and kW, respectively. Moreover, each irreducible character of $H_k(G, B)$ (resp. kW) is the specialization ψ_{f_q} (resp. ψ_{f_1}) of some irreducible character ψ of $A^{\bar{K}}$.*

2. Generic degrees of irreducible characters of $A(u)$. Let $A = A(u)$ be the generic algebra over D corresponding to (W, R). We know **[8**, Lemma 2.7] that there is a unique homomorphism ν: $A \to D$ satisfying $\nu(a_r) = u$ for all $r \in R$. Then $P(W) = \Sigma\{\nu(a_w): w \in W\}$ is called the *Poincaré polynomial*, and is a monic polynomial of degree $N = \max\{l(w): w \in W\}$.

If ψ is an irreducible character of $A^{\bar{K}}$ we set

$$d_\psi = \frac{P(W) \deg \psi}{\Sigma\{\nu(a_w)^{-1} \psi(\hat{a}_w) \psi(a_w): w \in W\}}, \tag{2.1}$$

where $\hat{a}_w = a_{w^{-1}}$, and call d_ψ the *generic degree* associated with ψ. Note that since $f_q P(W) = [G\colon B]$, then if $\tilde{\psi}_{f_q}$ is an irreducible constituent of 1_B^G whose restriction to $H_k(G, B)$ is ψ_{f_q}, then

$$f_q d_\psi = \tilde{\psi}_{f_q}(1),$$

by the Curtis-Fossum degree formula [7, Theorem 3.1]. Obviously,

$$f_1 d_\psi = \psi_{f_1}(1). \tag{2.2}$$

The following result from [1] is extremely important in the sequel.

THEOREM 2.3. *Suppose that the set $\mathcal{CP}$ of characteristic powers contains almost all primes. Then $d_\psi \in \mathbf{Q}[u]$ for each irreducible character ψ of $A(u)^{\bar{K}}$.*

This theorem is a strengthened version of that in [8, Theorem 5.7]. Note that this theorem does not apply to the Ree groups ${}^2G_2(3^{2m+1})$ and ${}^2F_4(2^{2m+1})$ or the Suzuki groups ${}^2B_2(2^{2m+1})$ (notation as in [5, p. 251]). The result is true, however, for twisted B_2 and G_2 since in these cases 1_B^G is a doubly transitive permutation character. The Ree groups ${}^2F_4(2^{2m+1})$ do provide a counterexample, for if π is the reflection representation of $A(u)$, as defined by R. Kilmoyer in [13], then

$$d_\pi = \frac{u^2(u+1)(u^2+1)(u^9+u^6+u^3+1)}{4(u+\sqrt{2u}+1)(u^3-u\sqrt{2u}+1)};$$

see [8, p. 111]. Theorem 2.3 does apply to the systems considered herein, viz., $G = E_6(q)$ and $G = E_7(q)$.

Let $J \subseteq R$ and let $W_J = \langle J \rangle$ be a parabolic subgroup of W. Then (W_J, J) is a finite Coxeter system. Let $A_J(u)$ be the corresponding generic algebra. Let ψ_J be an irreducible character of $A_J(u)^{\bar{K}}$ and let f_q and f_1 denote the usual specializations. Then ψ_{Jf_q} and ψ_{Jf_1} are irreducible linear characters of $H_k(G_J, B)$ and kW_J, respectively, where $G_J = BW_JB$. Let $\tilde{\psi}_{Jf_q}$ be the unique irreducible character of kG_J such that $\tilde{\psi}_{Jf_q} | H_k(G_J, B) = \psi_{Jf_q}$. Then the induced character $(\tilde{\psi}_{Jf_q})^G$ decomposes into irreducible characters of G. Because of (1.3) the restrictions of these characters to $H_k(G, B)$ are the specializations of irreducible characters ψ_i of $A(u)^{\bar{K}}$. Thus we may write

$$\left(\tilde{\psi}_{Jf_q}\right)^G = \sum m_i \tilde{\psi}_{if_q}. \tag{2.4}$$

Similarly we may decompose $(\psi_{Jf_1})^W$ into irreducible characters of W. Clearly

$$\left(\psi_{Jf_1}\right)^W = \sum m_i \psi_{if_1}. \tag{2.5}$$

If we set $P(W_J) = \Sigma\{\nu(a_w)\colon w \in W_J\}$ then $f_q P(W_J) = [G_J\colon B]$ and $f_1 P(W_J) = |W_J|$. Therefore

$$f_q(P(W)/P(W_J)) = [G\colon G_J], \quad f_1(P(W)/P(W_J)) = [W\colon W_J],$$

from which we conclude that

$$f_q\left(d_{\psi_J}P(W)/P(W_J)\right) = \left(\tilde{\psi}_{J,f_q}\right)^G(1),$$
$$f_1\left(d_{\psi_J}P(W)/P(W_J)\right) = \left(\psi_{J,f_1}\right)^W(1). \tag{2.6}$$

We remark that $P(W)/P(W_J)$ is in $\mathbf{Q}[u]$ and that it can be easily computed (see [**13**] or [**18**]).

Therefore we conclude that

$$d_{\psi_J}P(W)/P(W_J) = \sum m_i d_{\psi_i}, \tag{2.7}$$

since the expressions in (2.7) are in $\mathbf{Q}[u]$ and since (2.4) and (2.6) imply that (2.7) is simply a group theoretic fact for infinitely many specializations f_q: $u \mapsto q \in \mathcal{CP}$.

From (2.5) we see that the multiplicities m_i are obtained by inducing the irreducible character ψ_{J,f_1} to W and decomposing into irreducible characters of W. If the generic degree d_{ψ_J} is known, then (2.7) is a linear nonhomogeneous equation in the d_{ψ_i}. By performing the decomposition (2.5) for other irreducible characters of W_J we will, of course, obtain more equations as in (2.7), giving rise to a linear nonhomogeneous system in the generic degrees d_{ψ_i}. In case $G(q) = E_6(q)$ or $E_7(q)$, these systems are underdetermined, and we must obtain additional results to obtain unique solutions for the d_{ψ_i}.

Let w_0 be the longest word in W. Then $\nu(a_{w_0}) = u^N$, where N is the number of positive roots in the root system for W. Also, if $w \in W$ then $\nu(a_w) = u^{l(w)}$, and $l(w) < N$. Thus, from (1.3) we conclude that

$$\nu(a_{w_0})\sum\left\{\nu(a_w)^{-1}\psi(\hat{a}_w)\psi(a_w)\colon w \in W\right\} \in D^*, \tag{2.8}$$

for any irreducible character ψ of $A(u)^{\bar{K}}$. Since $d_\psi \in \mathbf{Q}[u]$, we conclude from (2.1) and (2.8) that

$$\nu(a_{w_0})P(W)\deg\psi/d_\psi \in D^* \cap K.$$

But it is a well-known fact from commutative algebra that $D^* \cap K = D = k[u]$ (see, e.g., [**14**, p. 240]). Therefore

$$d_\psi | \nu(a_{w_0})P(W) \tag{2.9}$$

in $\mathbf{Q}[u]$.

Equations (2.7) together with conditions (2.3) and (2.9) are entirely sufficient to determine uniquely the generic degrees of the constituents of 1_B^G when $G = E_6(q)$. These conditions almost suffice in case $G = E_7(q)$. However, if ψ_{f_1} is one of the irreducible characters of $W(E_7)$ of degree 512, an additional argument is necessary to compute d_ψ. With a few exceptions, the generic degrees of the constituents of 1_B^G, $G = E_8(q)$, are unknown.

3. Additional conditions on the generic degrees. Let $A = A(u)$ be the generic algebra of (W, R) and let σ be the linear character of A determined by $\sigma(a_r) = -1$, $r \in R$. Then if ψ is an irreducible character of $A^{\bar{K}}$, so is $\sigma\psi$. Moreover, in [10] J. A. Green showed that

$$d_{\sigma\psi} = \nu(a_{w_0})d_\psi(u^{-1}), \tag{3.1}$$

where w_0 is the longest word in W. If f_1: $u \mapsto 1$ then σ_{f_1} is the alternating character of W. Whenever there is no danger of confusion we write σ instead of σ_{f_1}.

Suppose W_J is a parabolic subgroup of W and that χ is an irreducible character of W_J. If

$$\chi^W = \sum m_i\chi_i, \tag{3.2}$$

where the χ_i are irreducible characters of W, then we write $\sigma\chi$ for the irreducible character $(\sigma | W_J)\chi$ of W_J and note that

$$(\sigma\chi)^W = \sigma\chi^W = \sum m_i\sigma\chi_i.$$

If χ is an irreducible character of W (or of some parabolic subgroup W_J) we write $d(\chi)$ for the generic degree of the corresponding irreducible character of $A(u)$ (or of $A_J(u)$). If χ_J is an irreducible character of W_J we write

$$d(\chi_J^W) = d(\chi_J)P(W)/P(W_J), \tag{3.3}$$

and note that, by (2.6), $d(\chi_J^W)$ specializes to the degree of the corresponding induced character of G (or of W). Since (3.1) relates $d(\chi)$ and $d(\sigma\chi)$ we need only compute the multiplicities in (3.2) corresponding to one irreducible character χ of W in each orbit under σ.

Assume for the time being, that W is one of the classical types A_n, $n > 1$, B_n, $n \geqslant 2$ or D_n, $n \geqslant 4$. The groups $W(A_n)$ are precisely the symmetric groups S_{n+1} on $n + 1$ letters, and the irreducible characters of $W(A_n)$ are in one-to-one correspondence with the partitions (α) of $n + 1$ (see [17] for notation). Moreover, if χ is an irreducible character of $W(A_n)$ with $\chi \leftrightarrow (\alpha)$, then it is well known that $\sigma\chi \leftrightarrow (\alpha^*)$, the dual partition of (α).

The groups $W(B_n)$ consist of the *signed* permutations on n letters and the irreducible characters of $W(B_n)$ are in one-to-one correspondence with the double partitions (α, β) of n (see [22]). If χ is an irreducible character of $W(B_n)$ with $\chi \leftrightarrow (\alpha, \beta)$, then $\sigma\chi \leftrightarrow (\beta^*, \alpha^*)$ (see [22]).

For fixed $n \geqslant 4$ the group $W(D_n)$ is the subgroup of index 2 in $W(B_n)$ consisting of signed permutations with an even number of minus signs. If χ is an irreducible character of $W(B_n)$ with $\chi \leftrightarrow (\alpha, \beta)$ then $\chi | W(D_n)$ is irreducible if and only if $(\alpha) \neq (\beta)$ (see [22]).

We now assume that W is of type E_6 or E_7 and that W_J is a maximal proper parabolic subgroup whose Dynkin diagram is connected. Thus, if

$W = W(E_6)$ then W_J is of type A_5 or D_5. If $W = W(E_7)$ then W_J is of type A_6, D_6 or E_6. If W_J and W_K are two parabolic subgroups of the same type in W, then, since the Dynkin diagram of (W, R) is simply laced, W_J and W_K are conjugate subgroups. Thus, inducing irreducible characters from W_J will yield the same decomposition multiplicities as inducing the corresponding characters of W_K. As a result, we need only consider one maximal parabolic subgroup of a given type. Moreover, it is unnecessary to specify exactly the subset $J \subseteq R$; it is only necessary to specify the type of subgroup.

If $P(W)$ is defined as in §2, then in [18] L. Solomon showed that

$$P(W) = \prod_i (1 + u + \cdots + u^{d_i - 1}),$$

where the d_i are the degrees of the polynomial invariants of W (see also [13]). For $W = W(E_6)$ the degrees of the invariants are 2, 5, 6, 8, 9 and 12, and for $W = W(E_7)$ the degrees of the invariants are 2, 6, 8, 10, 12, 14 and 18 (see [5, p. 155]). Thus, if $\Phi_0 = u$ and $\Phi_k = \Phi_k(u)$ is the kth cyclotomic polynomial over $\mathbf{Q}$, then we may factor $P(W)$ as

$$P(W) = \begin{cases} \Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}, & \text{if } W = W(E_6), \\ \Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}, & \text{if } W = W(E_7). \end{cases}$$

It is well known that if w_0 is the longest word of W, then $\nu(a_{w_0}) = u^{36}$, if $W = W(E_6)$, and $\nu(a_{w_0}) = u^{63}$ if $W = W(E_7)$. Therefore, we conclude from (2.9) that for each irreducible character ψ of $W(E_6)$, $d(\psi) | u^{36}P(E_6)$. Since $d(\psi) \in \mathbf{Q}[u]$, we may therefore write

$$d(\psi) = a \prod \Phi_i^{k_i}, \qquad a \in \mathbf{Q}, \tag{3.4}$$

where, for $i = 2, 3, 4, 5, 6, 8, 9$ and 12,

$$0 \leqslant k_i \leqslant h_i \quad \text{and} \quad 0 \leqslant k_0 \leqslant 36, \tag{3.5}$$

where $P(E_6) = \prod \Phi_i^{h_i}$ as in (4.2). Obviously, similar statements hold for $d(\psi)$ in case $W = W(E_7)$. Thus, the determination of the generic degrees $d(\psi)$ is tantamount to finding the coefficient a and the exponents k_i.

If f_1 is the usual specialization then from (2.2) and (3.4) we conclude that

$$\psi(1) = a \prod f_1 \Phi_i^{k_i}. \tag{3.6}$$

If the coefficient a is known then (3.6) gives linear relations on the k_i. For example, it can be shown that for the irreducible character $\psi = 30_p$ of $W(E_6)$ the leading coefficient of $d(30_p)$ is $a = 1/2$. Thus, since $30_p(1) = 30 = 2 \cdot 3 \cdot 5$, and since $f_1\Phi_i = 2, 3, 2, 5, 1, 2, 3$ and 1, for $i = 2, 3, 4, 5, 6, 8, 9$ and 12, respectively, we conclude from (3.6) that $k_2 + k_4 + k_8 = 2$, $k_3 + k_9 = 1$ and that $k_5 = 1$. (3.6) provides no information about k_6 or k_{12}.

If $\deg d(\psi) = d$ is known, then we have another linear equation in the k_i, namely

$$d = \sum k_i \cdot \deg \Phi_i. \tag{3.7}$$

Finally, since $f_1(\Phi_i) \geqslant 1$ for all i, we conclude that $a > 0$.

In summary, (2.7), (2.9) and (3.1)–(3.7) turn out to be sufficient for the determination of the generic degrees $d(\psi)$ ($\deg \psi \neq 512$) in case $W = W(E_6)$ or $W = W(E_7)$. The actual application of the above equations and conditions is rather ad hoc in nature and so the details will be suppressed. For a more explicit version of the computations, see [20].

We shall conclude this section with the additional argument required to compute $d(512_a)$, where 512_a is the character of degree 512 of $W(E_7)$ as in [9]. Let ψ be an irreducible representation of $A(u)^{\overline{K}}$ of degree 512 such that ψ specializes to the character 512_a of $W(E_7)$. In [6] it is shown that $\det \psi(a_{w_0}) = u^{63/2}$. Since a_{w_0} is central in $A(u)^{\overline{K}}$, $\psi(a_{w_0})$ is a scalar matrix. Thus

$$\psi(a_{w_0}) = \operatorname{diag}(\sqrt{u}\,, \sqrt{u}\,, \ldots, \sqrt{u}\,). \tag{3.8}$$

Now let L be a finite normal extension of $\mathbf{Q}[u]$ which contains all entries of every $\psi(a_w)$, $w \in W(E_7)$, and let δ be an automorphism of L such that $\delta\sqrt{u} = -\sqrt{u}$. Then the representation ψ^δ of $A(u)^{\overline{K}}$ is an irreducible representation of degree 512 and, from (3.8), is distinct from ψ. From (2.1) it is clear that $\delta d_\psi = d_{\psi^\delta}$, but since $d_\psi \in \mathbf{Q}[u]$, we conclude that $d_\psi = d_{\psi^\delta}$. By using the methods outlined in this section it can be shown that

$$d(512_a) + d(512_{-a}) = \Phi_0^{11}\Phi_2^7\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18},$$

and so $d(512_a) = \frac{1}{2}\Phi_0^{11}\Phi_2^7\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$.

4. Generic degrees corresponding to irreducible characters of $W = W(E_6)$ and $W = W(E_7)$. In this section we give, in Tables 1 and 2, the generic

TABLE 1. Generic degrees of irreducible characters of $W(E_6)$

Characters	Generic Degrees	Characters	Generic Degrees
1_p	1	24_p	$\Phi_0^6\Phi_4^2\Phi_8\Phi_9\Phi_{12}$
6_p	$\Phi_0\Phi_8\Phi_9$	60_p	$\Phi_0^5\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{12}$
15_p	$\frac{1}{2}\Phi_0^3\Phi_5\Phi_6^2\Phi_8\Phi_9$	20_s	$\frac{1}{6}\Phi_0^7\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9$
20_p	$\Phi_0^2\Phi_4\Phi_5\Phi_8\Phi_{12}$	90_s	$\frac{1}{3}\Phi_0^7\Phi_3^3\Phi_5\Phi_6^2\Phi_8\Phi_{12}$
30_p	$\frac{1}{2}\Phi_0^3\Phi_4^2\Phi_5\Phi_9\Phi_{12}$	80_s	$\frac{1}{6}\Phi_0^7\Phi_2^4\Phi_5\Phi_8\Phi_9\Phi_{12}$
64_p	$\Phi_0^4\Phi_2^3\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$	60_s	$\frac{1}{2}\Phi_0^7\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}$
81_p	$\Phi_0^6\Phi_3^3\Phi_6^2\Phi_9\Phi_{12}$	10_s	$\frac{1}{3}\Phi_0^7\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$
15_q	$\frac{1}{2}\Phi_0^3\Phi_5\Phi_8\Phi_9\Phi_{12}$		

TABLE 2

Generic degrees of irreducible characters of $W(E_7)$

Characters	Generic Degrees	Characters	Generic Degrees
1_a	1	405_a	$\frac{1}{2}\Phi_0^8\Phi_3^3\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$
7_a	$\Phi_0^{46}\Phi_7\Phi_{12}\Phi_{14}$	168_a	$\Phi_0^6\Phi_4^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$
27_a	$\Phi_0^2\Phi_3^2\Phi_6^2\Phi_9\Phi_{12}\Phi_{18}$	56_a	$\frac{1}{2}\Phi_0^{30}\Phi_2^4\Phi_6^2\Phi_7\Phi_{10}\Phi_{14}\Phi_{18}$
21_a	$\frac{1}{2}\Phi_0^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	120_a	$\frac{1}{2}\Phi_0^4\Phi_2^4\Phi_5\Phi_6^2\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$
35_a	$\frac{1}{6}\Phi_0^{16}\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	210_a	$\Phi_0^6\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$
105_a	$\frac{1}{2}\Phi_0^{25}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	280_a	$\frac{1}{3}\Phi_0^{16}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$
189_a	$\frac{1}{2}\Phi_0^8\Phi_3^2\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	336_a	$\frac{1}{2}\Phi_0^{13}\Phi_2^4\Phi_6^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$
21_b	$\Phi_0^{36}\Phi_7\Phi_9\Phi_{14}\Phi_{18}$	216_a	$\frac{1}{2}\Phi_0^{15}\Phi_2^4\Phi_3^2\Phi_6^3\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
35_b	$\frac{1}{2}\Phi_0^3\Phi_5\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{18}$	512_a	$\frac{1}{2}\Phi_0^{11}\Phi_2^7\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
189_b	$\Phi_0^{22}\Phi_3^2\Phi_6^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	378_a	$\Phi_0^{14}\Phi_3^2\Phi_6^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$
189_c	$\Phi_0^{20}\Phi_3^2\Phi_6^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	84_a	$\frac{1}{2}\Phi_0^{10}\Phi_4^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
15_a	$\frac{1}{2}\Phi_0^{25}\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	420_a	$\frac{1}{2}\Phi_0^{10}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$
105_b	$\Phi_0^6\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	280_b	$\frac{1}{2}\Phi_0^7\Phi_2^4\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
105_c	$\Phi_0^{12}\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	210_b	$\Phi_0^{10}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
315_a	$\frac{1}{6}\Phi_0^{16}\Phi_3^3\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	70_a	$\frac{1}{3}\Phi_0^{16}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$

degree of each irreducible character of W in a given orbit under σ, where $W = W(E_6)$ and $W = W(E_7)$, respectively.

5. Appendix. *Miscellaneous tables.* In this Appendix we give the decomposition multiplicities and generic degrees of (2.7) corresponding to selected irreducible characters of parabolic subgroups of $W(E_6)$ and $W(E_7)$.

To compute the decomposition multiplicities for induced characters from W_J, we need the character tables for the groups W and the subgroups W_J. For $W_J = W(A_n)$, $n = 5$ or 6, these tables can be found, e.g., in [15]. If $W_J = W(D_n)$ then W_J is a semidirect product of S_n and an elementary abelian normal subgroup of order 2^{n-1}. One can then use a procedure given by Mackey in [16] to construct the character tables. Finally, the character tables of $W(E_6)$ and $W(E_7)$ were computed by Frame in [9].

TABLE 3

Decomposition of irreducible characters of

$W_J = W(A_5)$ induced to $W = W(E_6)$

$W_J \backslash W$	1_p	6_p	15_p	20_p	30_p	64_p	81_p	15_q	24_p	60_p	20_s	90_s	80_s	60_s	10_s
[6]	1	1		1	1			1							
[51]		1	1	2	1	2	1			1					
[42]				1	1	2	1	1	1	2		1	1	1	
$[42^2]$			1		1	2/1	2		1		1	2	1		
$[3^2]$					1		1		0/1	1			1	1	1
[321]						1/1	2/2			2/2		2	2	2	

TABLE 4

Decomposition of irreducible characters of

$W_J = W(D_5)$ induced to $W = W(E_6)$

$W_J \backslash W$	1_p	6_p	15_p	20_p	30_p	64_p	81_p	15_q	24_p	60_p	20_s	90_s	80_s	60_s	10_s
[5],[0]	1	1		1											
[41],[0]				1		1			1						
[32],[0]								1		1				1	
$[41^2]$,[0]							1/1								
[4],[1]		1	1	1	1	1									
[31],[1]					1	1	1			1		1	1		
$[2^2]$,[1]										1/1			1	1	1
[3],[2]				1	1	1	1	1		1					
[21],[2]						1	1/1		1	1		1	1	1	
[3],$[1^2]$			1			1	1				1	1			

TABLE 5

Decomposition of irreducible characters of $W_J = W(A_6)$ induced to $W = W(E_7)$

$W_J \backslash W$	1_a	7_a	27_a	21_a	35_a	105_a	189_a	21_b	35_b	189_b	189_c	15_a	105_b	105_c	315_a	405_a
[7]	1	0/1	1					0/1	1	0/1		0/1	1			
[61]		0/1	2	1		0/2		0/1	1	0/2	0/1		1		0/1	1
[52]			1			0/2	1	0/1	1	0/3	0/2		1		0/2	2
$[51^2]$				1	0/1	0/2	2			0/1	0/2			1	0/2	2
[43]									1	0/2	0/1	0/1	2	1	0/2	2
[421]						0/1	2/1			0/2	0/2		1	0/1	1/3	4/2
$[3^21]$										0/1	1		1	1	1/2	2/1
$[41^3]$					1/1		2/2				1/1			1/1	1/1	2/2

TABLE 5, *continued*

W_J \ W	168_a	56_a	120_a	210_a	280_a	336_a	216_a	512_a	378_a	84_a	420_a	280_b	210_b	70_a
[7]		0/1	1											
[61]	1	0/2	2	2	0/1		0/1					1		
[52]	3	0/1	2	2	0/1	0/1	0/1	1/1	0/2		1	2	1	0/1
[51^2]	1	0/1	2	3	0/3	1/2		1/1	0/1		2/1	1		
[43]	1	0/1	1	1	0/1		0/2	1/1	1/2	1/1	1/1	2	2/1	0/1
[421]	2		1	2	1/3	2/3	1/2	4/4	2/4	1	4/2	3/1	1/1	
[3^21]	0/1			1	0/1	1/1	1/2	3/3	2/1	0/1	1/2	2/1	2/2	1/1
[41^3]				1/1	2/2	3/3		1/1	1/1		3/3			

TABLE 6

Decomposition of irreducible characters of $W_J = W(D_6)$ induced to $W = W(E_7)$

W_J \ W	1_a	7_a	27_a	21_a	35_a	105_a	189_a	21_b	35_b	189_b	189_c	15_a	105_b	105_c	315_a	405_a
[6],[0]	1	0/1	1						1							
[51],[0]			1			0/1		0/1			0/1					
[42],[0]									1	0/1						
[41^2],[0]											0/1			0/1		
[3^2],[0]												0/1	1			
[321],[0]																
[5],[1]		0/1	1	1		0/1		0/1		0/1						
[41],[1]						0/1	1			0/1	0/1				0/1	1
[32],[1]										0/1			1		0/1	1
[31^2],[1]															1/1	1/1
[4],[2]			1			0/1			1	0/1			1		0/1	
[31],[2]										0/1	0/1		1		0/1	2
[2^2],[2]															0/1	0/1
[21^2],[2]							1/1							0/1	1	1/1
[4],[1^2]				1	0/1	0/1	1				0/1					1
[21],[3]						0/1	1			0/1	0/1				0/1	1

If χ is an irreducible character of $W(A_n)$ and $\chi \leftrightarrow (\alpha) = (\alpha_1, \ldots, \alpha_k)$ we shall write $[\alpha_{i_1}^{k_1} \cdots \alpha_{i_m}^{k_m}]$ for χ. For example, [321] will denote the character corresponding to the partition (3, 2, 1) of 6. Similarly, [2^31] will denote the character corresponding to the partition (2, 2, 2, 1) of 7. We shall use similar notations for irreducible characters of $W(D_n)$.

If χ is an irreducible character of $W(E_6)$ or $W(E_7)$ we shall use the notation in [9] to denote χ. We mention here that $W(E_7) = \Theta \times \langle w_0 \rangle$ where Θ is the rotation subgroup of $W(E_7)$ and w_0 is the longest word in $W(E_7)$. In [9] Frame computed the thirty irreducible characters of Θ. We shall use the same notations to denote the corresponding characters of $W(E_7)$ obtained by

TABLE 6, *continued*

W / W_J	168_a	56_a	120_a	210_a	280_a	336_a	216_a	512_a	378_a	84_a	420_a	280_b	210_b	70_a
[6],[0]		0/1												
[51],[0]	1		1											
[42],[0]	1								0/1	1		1		
$[41^2]$,[0]				1		0/1					1			
$[3^2]$,[0]							0/1			0/1			1	
[321],[0]							1/1	1/1				1/1		
[5],[1]		0/1	1	1										
[41],[1]	1	0/1	1	1	0/1				0/1		1			
[32],[1]							0/1	1/1	1/1			1	1/1	0/1
$[31^2]$,[1]					1/1	1/1		1/1			1/1			
[4],[2]	1	0/1	1	1	0/1							1		
[31],[2]	1		1	1	0/1	0/1	0/1	1/1	0/1		1/1	1	1	
$[2^2]$,[2]	1						1	1/1	0/1	1	1	1	1/1	0/1
$[21^2]$,[2]				0/1	1	1/1		1/1	1/1		1/1	0/1		
[4],$[1^2]$			1	1	0/1	0/1								
[21],[3]	1			1	0/1	1/1	0/1	1/1	0/1		1	1		

TABLE 7

Decomposition of irreducible characters of $W_J = W(E_6)$ induced to $W = W(E_7)$

W / W_J	1_a	7_a	27_a	21_a	35_a	105_a	189_a	21_b	35_b	189_b	189_c	15_a	105_b	105_c	315_a	405_a
1_p	1	0/1	1					0/1								
6_p		0/1	1	1		0/1										
15_p				1	0/1	0/1	1									
20_p			1			0/1		0/1	1	0/1	0/1					
30_p										0/1			1		0/1	1
64_p						0/1	1			0/1	0/1				0/1	1
81_p											0/1			1	0/1	1
15_q									1	0/1		0/1	1			
24_p											0/1			0/1		
60_p										0/1			1		0/1	1
20_s					1/1		1/1									
90_s							1/1									1/1
80_s															1/1	1/1
60_s																
10_s																

extending trivially across $\langle w_0 \rangle$. The remaining thirty irreducible characters can be obtained by multiplying the original character values by -1 on the odd classes. Finally, if N_m denotes one of the irreducible characters of degree N of $W(E_6)$ or $W(E_7)$, we shall usually write N_{-m} in place of σN_m. The

TABLE 7, *continued*

W_J \ W	168_a	56_a	120_a	210_a	280_a	336_a	216_a	512_a	378_a	84_a	420_a	280_b	210_b	70_a
1_p														
6_p		0/1	1											
15_p				1	0/1									
20_p	1	0/1	1	1										
30_p		0/1	1	1	0/1									
64_p	1		1	1	0/1	0/1					1	1		
81_p				1	0/1	1/1	0/1	1/1	0/1		1/1	1		
15_q							0/1					1		
24_p	1								0/1	1	1			
60_p	1						0/1	1/1	1			1	1	0/1
20_s						1/1								
90_s					1/1	1/1		1/1	1/1		1/1			
80_s								1/1	1/1		1/1		1/1	
60_s							1/1	1/1	1/1	1/1		1/1	1/1	
10_s													1/1	1/1

irreducible characters N_s of $W(E_6)$ are "self-associated" characters in that $N_s = N_{-s}$.

Each row in Tables 3–7 gives the decomposition multiplicities for a character induced from W_J. The notation h/k in the column corresponding to the character ζ of W and in the row corresponding to the character χ of W_J means that

$$(\chi^W, \zeta) = h$$

and

$$(\chi^W, \sigma\zeta) = k.$$

The omitted entries in the tables are understood to be 0's.

With the exception of $W_J = W(D_6) < W(E_7)$ in Table 6, we have given, for each irreducible character χ of W_J, the decomposition of χ^W (or of $\sigma\chi^W$) into irreducible characters of W. The irreducible characters omitted in case $W_J = W(D_6)$ are not necessary for the computations of the generic degrees of constituents of 1_B^G where $W = W(E_7)$.

We have used formulas given by Hoefsmit in [11] to compute the generic degrees corresponding to the irreducible characters of W_J in Tables 3–6 and list the generic degrees in Table 8.

ADDED IN PROOF. The author has learned that C. T. Benson has computed the generic degrees d in case $G = E_8(q)$ in C. T. Benson, *The generic degrees of the irreducible characters of* E_8 (to appear).

TABLE 8

Generic degrees of selected irreducible characters of $W(A_5)$, $W(D_5)$, $W(A_6)$ and $W(D_6)$.

$\Phi_0 = u$ and $\Phi_k = \Phi_k(u)$ is the kth cyclotomic polynomial.

Characters	Generic Degrees	Characters	Generic Degrees
$W(A_5)$:		$W(A_6)$:	
[6]	1	[7]	1
[51]	$\Phi_0\Phi_5$	[61]	$\Phi_0\Phi_2\Phi_3\Phi_6$
[42]	$\Phi_0^2\Phi_3^2\Phi_6$	[52]	$\Phi_0^2\Phi_4\Phi_7$
$[41^2]$	$\Phi_0^3\Phi_4\Phi_5$	$[51^2]$	$\Phi_0^3\Phi_3\Phi_5\Phi_6$
$[3^2]$	$\Phi_0^3\Phi_5\Phi_6$	[43]	$\Phi_0^3\Phi_2\Phi_6\Phi_7$
[321]	$\Phi_0^4\Phi_2^3\Phi_4\Phi_6$	[421]	$\Phi_0^4\Phi_5\Phi_7$
		$[3^21]$	$\Phi_0^5\Phi_3\Phi_6\Phi_7$
		$[41^3]$	$\Phi_0^6\Phi_2\Phi_4\Phi_5\Phi_6$
$W(D_5)$:		$W(D_6)$:	
[5],[0]	1	[6],[0]	1
[41],[0]	$\Phi_0^2\Phi_4\Phi_8$	[51],[0]	$\Phi_0^2\Phi_5\Phi_{10}$
[32],[0]	$\frac{1}{2}\Phi_0^3\Phi_5\Phi_6\Phi_8$	[42],[0]	$\frac{1}{2}\Phi_0^3\Phi_3^2\Phi_6\Phi_8\Phi_{10}$
$[31^2]$,[0]	$\Phi_0^6\Phi_3\Phi_6\Phi_8$	$[41^2]$,[0]	$\Phi_0^6\Phi_5\Phi_8\Phi_{10}$
[4],[1]	$\Phi_0\Phi_5\Phi_6$	$[3^2]$,[0]	$\frac{1}{2}\Phi_0^4\Phi_5\Phi_6^2\Phi_8\Phi_{10}$
[31],[1]	$\frac{1}{2}\Phi_0^3\Phi_3\Phi_5\Phi_8$	[321],[0]	$\frac{1}{2}\Phi_0^7\Phi_2^4\Phi_3\Phi_6^3\Phi_8$
$[2^2]$,[1]	$\Phi_0^5\Phi_5\Phi_6\Phi_8$	[5],[1]	$\Phi_0\Phi_3\Phi_6\Phi_8$
[3],[2]	$\Phi_0^2\Phi_5\Phi_8$	[41],[1]	$\frac{1}{2}\Phi_0^3\Phi_2^4\Phi_3^2\Phi_6^3$
[21],[2]	$\Phi_0^4\Phi_4\Phi_5\Phi_8$	[32],[1]	$\Phi_0^5\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}$
[3],$[1^2]$	$\frac{1}{2}\Phi_0^3\Phi_4^2\Phi_5\Phi_6$	$[31^2]$,[1]	$\frac{1}{2}\Phi_0^7\Phi_3^2\Phi_4^2\Phi_8\Phi_{10}$
		[4],[2]	$\Phi_0^2\Phi_3\Phi_5\Phi_6\Phi_{10}$
		[31],[2]	$\frac{1}{2}\Phi_0^4\Phi_3^2\Phi_5\Phi_8\Phi_{10}$
		$[2^2]$,[2]	$\Phi_0^6\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}$
		$[21^2]$,[2]	$\Phi_0^8\Phi_3^2\Phi_5\Phi_6^2\Phi_{10}$
		[4],$[1^2]$	$\frac{1}{2}\Phi_0^3\Phi_3\Phi_5\Phi_6^2\Phi_8$
		[21],[3]	$\frac{1}{2}\Phi_0^4\Phi_2^4\Phi_5\Phi_6^2\Phi_{10}$

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